

用行列修正拟 Newton 法求解奇异问题

王 颖, 潘状元

(哈尔滨理工大学 应用科学学院, 黑龙江 哈尔滨 150080)

摘 要: 给出了求解奇异问题的行列修正拟 Newton 法的收敛性定理及渐近收敛速率.

关键词: 拟 Newton 法; 奇异问题; 收敛速率

中图分类号: O242. 23 **文献标志码:** A **文章编号:** 1007-2683(2010)04-0064-04

Using Row-column-update Quasi-newton Methods to Solve the Singular Problems

WANG Ying, PAN Zhuang-yuan

(School of Applied Science, Harbin University of Science and Technology, Harbin 150080, China)

Abstract: In this paper, using the convergence of row-column-update quasi-Newton to solve singular problems equations is proven and the asymptotic convergence rate is obtained.

Key words: quasi-newton method; singular problem; asymptotic convergence rate

假设 F 是 $\mathbf{R}^n \rightarrow \mathbf{R}^n$ 的可微映射, $x^* \in \mathbf{R}^n$ 是 $F(x) = 0$ 的解, 且 $F'(x^*)$ 奇异.

我们考虑如下的行列修正拟牛顿法:

$$x_{k+1} = x_k - E_k^{-1} F(x_k) \quad (1)$$

$$E_{k+1} = (I - e_{i_{k+1}} e_{i_{k+1}}^T) E_k (I - e_{i_{k+1}} e_{i_{k+1}}^T) + \frac{e_{i_{k+1}} y_{k+1}^T + y_{k+1} e_{i_{k+1}}^T}{\|S_{k+1}\|} - \frac{(y_{k+1}, e_{i_{k+1}}) e_{i_{k+1}} e_{i_{k+1}}^T}{\|S_{k+1}\|}$$

$$i_k = k \pmod{n} \quad (2)$$

$$S_k = x_{k+1} - x_k, y_k = F(x_k) - F(x_k - \|S_{k-1}\| e_{i_k}) \quad (3)$$

其中 $x_{k-1}, x_k \in \mathbf{R}^n, E_{k-1}, E_k \in L(\mathbf{R}^n), I \in L(\mathbf{R}^n)$ 是单位矩阵, $e_{i_k} \in \mathbf{R}^n$ 是单位坐标向量.

本文始终假定 $F \in C^3$, 且 $F'(x^*)$ 的零空间满足:

$$N = \text{span}\{\phi\}, \|\phi\| = \langle \phi, \phi \rangle^{\frac{1}{2}} = 1$$

且

$$\phi \in \mathbf{R}^n, \mathbf{R}^n = N \oplus X \quad (4)$$

$$P_N: \mathbf{R}^n \rightarrow N, P_X = I - P_N$$

$$P_N F''(x^*)(\phi, \phi) \neq 0 \quad (5)$$

对于 $\rho > 0, \theta > 0$ 及整数 $v \geq 1$, 定义: $W(\rho, \theta, v) = \{x \mid x \in \mathbf{R}^n, 0 < \|\tilde{x}\| \leq \rho, \|P_X \tilde{x}\| \leq \theta \|P_N \tilde{x}\|^v\}$

记 $\tilde{x} = x - x^*, \beta_l(x)$ 是与 $\|x - x^*\|^l$ 同阶的无穷小量.

不失一般性, 我们假定 $F'(x^*)$ 是对称的, 且 $X = N^\perp$. 假若不然, 我们可以找到一个非奇异映射 L 使得 $LF'(x^*)$ 是对称的, 此时, 只需考虑等价方程 $LF(x) = 0$, 但算法式(1)-(3)保持不变.

假设 $\phi = e_n$, 其中 $e_n = (0, \dots, 0, 1)^T$, 即子空间 N 与坐标轴 X_n 重合.

对于矩阵 $E_n \in \mathbf{R}^n \times \mathbf{R}^n$, 如同文[1]定义:

$$\begin{aligned} A_n &= P_X E_n P_X & B_n &= P_X E_n P_N \\ C_n &= P_N E_n P_X & D_n &= P_N E_n P_N \end{aligned} \quad (6)$$

由于 A_n 作为 $X \rightarrow X$ 的映射是非奇异的, 我们

收稿日期: 2008-06-26

基金项目: 国家自然科学基金(19971022)

作者简介: 王 颖(1982—), 女, 硕士研究生, E-mail: wangyingdubiao@sina.com;

潘状元(1954—), 男, 教授, 硕士生导师.

定义:

$$\tilde{D}_n = D_n - C_n A_n^{-1} B_n \quad (7)$$

则 E_n 非奇异的充分必要条件是 $\tilde{D}_n$ 是 N 上的非奇异映射,且有

$$\begin{aligned} E_n^{-1} = & P_X (A_n^{-1} + A_n^{-1} B_n \tilde{D}_n^{-1} C_n A_n^{-1}) P_X - \\ & P_X A_n^{-1} B_n \tilde{D}_n^{-1} P_N - P_N \tilde{D}_n^{-1} C_n A_n^{-1} P_X + \\ & P_N \tilde{D}_n^{-1} P_N \end{aligned} \quad (8)$$

为了证明定理的需要,不妨假设 $F \in C^3$,可以证明存在闭球

$$S(x^*, \delta) = \{x \mid \|x - x^*\| < \delta, \delta > 0\} \subset \mathbb{R}^n$$

对于充分小的 ρ, θ 满足

$$\begin{aligned} P_X \tilde{x}_1 &= \theta \beta_2(x_0) + \beta_3(x_0) \\ P_N \tilde{x}_1 &= \lambda_0 \rho \phi \quad \frac{3}{8} < \lambda_0 < \frac{4}{5} \end{aligned} \quad (9)$$

因此存在一个充分小的 θ_1 ,使得

$$x_1 \in W(\rho, \theta_1, 2)$$

1 主要结果

定理 1 设 $x_0 \in W(\rho, \theta_0, 1)$, 假设式(4)(5)成立,并且 E_0^{-1} 存在 $x_1 \in W(\rho, \theta_1, 2)$, 又设

$$\|E_0 - F'(x^*)\| \leq k\rho = \varepsilon_0$$

其中 k 为给定的正数并且设

$$\begin{aligned} P_N \tilde{x}_0 &= \rho_0 \phi, P_N \tilde{x}_1 = \rho_1 \phi \\ \lambda_0 &= \frac{\rho_1}{\rho_0} \in \left(\frac{3}{8}, \frac{4}{5}\right) \end{aligned}$$

则对于充分小的 ρ, θ 由式(1) - (3)产生的序列 $\{x_k\}$ 收敛于 x^* 且有如下的敛速估计

$$\lim_{k \rightarrow \infty} \frac{\|x_{k+1} - x^*\|}{\|x_k - x^*\|} = \frac{\sqrt{5} - 1}{2} \quad (10)$$

$$\lim_{k \rightarrow \infty} \frac{\|P_X \tilde{x}_{k+1}\|}{\|P_N \tilde{x}_k\|} = 0 \quad (11)$$

证明: 设 $\theta_0 = \frac{\theta}{\rho_0}, x_0 \in W(\rho, \theta_0, 2)$ 我们用数学归纳法证明此定理,即对于所有的非负整数 $k \geq 0$,以下的关系式成立: $x_{k+1} \in W(\rho, \theta_{k+1}, 2)$

$$\left. \begin{aligned} \theta_{k+1} &\leq c_2(\varepsilon_{k-1}\theta_{k-1} + (1 + \varepsilon_k)\theta_k + \theta_{k-1}), \\ i_{k+1} &\neq n \\ \theta_{k+1} &\leq c_2(\varepsilon_{k+1}\theta_{k+1} + \varepsilon_k\tilde{\theta}_k), \\ i_{k+1} &= n \end{aligned} \right\} \quad (12)$$

其中 $\tilde{\theta}_k = \theta_{k+1} + \theta_k$ $c_2 > 0, k \geq 1$

$$P_N \tilde{x}_{k+1} = \rho_{k+1} \phi = \lambda_k \rho_k \phi = \lambda_k P_N \tilde{x}_k \quad (13)$$

$$\lambda_k \in \left(\frac{3}{8}, \frac{4}{5}\right)$$

$$\|E_k - \hat{F}\| \leq \varepsilon_0 + c_1 \rho_0 \sum_{p=1}^k \left(\frac{4}{5}\right)^p = \varepsilon_k \quad (14)$$

其中 $c_1 > 0$

$$\|A_k^{-1}\| \leq 2 \|\hat{F}^{-1}\| \quad \text{其中 } \hat{F} = F'(x^*),$$

$$\hat{F}^{-1} \text{ 是 } \hat{F}|_X \text{ 的逆} \quad (15)$$

当 $k=0$, (12) ~ (15) 显然成立,现在假设当 $k \leq n$ 时(12) ~ (15) 也成立,下面证明当 $k=n+1$ 时上述关系式也成立.为了证明上述结果我们给出如下重要引理.

引理 1 设定理 1 的假设成立,且(12) ~ (15) 当 $k \leq n$ 时成立,则对于 $k=n+1$ 时关系式(14), (15) 及 $\|c_{k+1}\| \leq \varepsilon_{k+1}$ 成立.

证明: 由式(2)和(3)得

$$\begin{aligned} E_{k+1} - \hat{F} = & (I - e_{i_{k+1}} e_{i_{k+1}}^T) (E_k - \hat{F}) (I - e_{i_{k+1}} e_{i_{k+1}}^T) - \\ & \hat{F} e_{i_{k+1}} e_{i_{k+1}}^T - e_{i_{k+1}} e_{i_{k+1}}^T \hat{F} - e_{i_{k+1}} e_{i_{k+1}}^T \hat{F} e_{i_{k+1}} e_{i_{k+1}}^T + \\ & \frac{e_{i_{k+1}} y_{k+1}^T + y_{k+1} e_{i_{k+1}}^T}{\|S_k\|} - \frac{(y_{k+1}, e_{i_{k+1}}^T)}{\|S_k\|} e_{i_{k+1}} e_{i_{k+1}}^T \end{aligned}$$

$$\begin{aligned} \|E_{k+1} - \hat{F}\| \leq & \|(I - e_{i_{k+1}} e_{i_{k+1}}^T) (E_k - \hat{F}) (I - \\ & e_{i_{k+1}} e_{i_{k+1}}^T)\| + \|\hat{F} e_{i_{k+1}} e_{i_{k+1}}^T + \\ & e_{i_{k+1}} e_{i_{k+1}}^T \hat{F} + e_{i_{k+1}} e_{i_{k+1}}^T \hat{F} e_{i_{k+1}} e_{i_{k+1}}^T\| + \\ & \frac{\|e_{i_{k+1}} y_{k+1}^T + y_{k+1} e_{i_{k+1}}^T\|}{\|S_k\|} - \\ & \frac{\|(y_{k+1}, e_{i_{k+1}}^T)\|}{\|S_k\|} \|e_{i_{k+1}} e_{i_{k+1}}^T\| \leq \\ & \varepsilon_k + c_1 \rho_{k+1} = \varepsilon_k + c_1 \lambda_k \rho_k \leq \\ & \varepsilon_k + c_1 \rho_0 \left(\frac{4}{5}\right)^{k+1} = \varepsilon_{k+1} \end{aligned} \quad (16)$$

式(16)表明对于 $k=n+1$ 时式(14)、(15)成立.

由于 $c_{k+1} = P_N E_{k+1} P_X = P_N (E_{k+1} - \hat{F}) P_X$ 所以

$$\begin{aligned} \|c_{k+1}\| &\leq \|P_N\| \|E_{k+1} - \hat{F}\| \|P_X\| = \\ &\|E_{k+1} - \hat{F}\| \leq \varepsilon_{k+1} \end{aligned}$$

所以 $\|c_{k+1}\| \leq \varepsilon_{k+1}$ 也成立.

引理 2 设引理 1 的假设成立,则

$$\left\{ \begin{aligned} D_{k+1} \phi &= \frac{1}{2} \rho_k (1 + \lambda_k) \langle u, \phi \rangle \phi + \theta_k \beta_0(x_{k+1}) + \\ & (1 + \tilde{\theta}_k) \beta_2(x_{k+1}) \quad e_{i_{k+1}} = \phi \\ D_{k+1} \phi &= \frac{\rho_k}{2(1 - \lambda_k)} \langle u, \phi \rangle \phi + \varepsilon_k \tilde{\theta}_k \beta_1(x_{k+1}) + \\ & (1 + \theta_k) \beta_2(x_{k+1}) \quad e_{i_{k+1}} \neq \phi \end{aligned} \right.$$

$$\begin{cases} B_{k+1}\phi = \frac{1}{2}\rho_k(1+\lambda_k)P_X u + \theta_k\beta_0(x_{k+1}) + \\ (1+\tilde{\theta}_k)\beta_2(x_{k+1}) & e_{i_{k+1}} = \phi \\ B_{k+1}\phi = \frac{\rho_k}{2(1-\lambda_k)}P_X u + (1+\theta_k)\beta_2(x_{k+1}) + \\ (\varepsilon_k\tilde{\theta}_k + \theta_{k+1})\beta_1(x_{k+1}) & e_{i_{k+1}} \neq \phi \end{cases}$$

其中 $u = F''(x^*)(\phi, \phi)$

证明:

$$F(x_k) = \hat{F}x_k + \frac{1}{2}F''(x^*)(\tilde{x}_k, \tilde{x}_k) + \beta_3(x_k) =$$

$$\frac{1}{2}\rho_k^2 u + \hat{F}\tilde{x}_k + (1+\theta_k)\beta_3(x_k)$$

$$S_k = \tilde{x}_{k+1} - \tilde{x}_k = (\lambda_k - 1)P_N\tilde{x}_k + P_X S_k =$$

$$(\lambda_k - 1)\rho_k\phi + P_X S_k$$

因为 $E_k S_k = -F(x_k)$

$$E_k S_k = (\lambda_k - 1)\rho_k E_k \phi + E_k P_X S_k =$$

$$-\frac{1}{2}\rho_k^2 u - \hat{F}\tilde{x}_k + (1+\theta_k)\beta_3(x_k) \quad (17)$$

由式(17)得

$$E_k \phi = \frac{\rho_k u}{2(1-\lambda_k)} + \frac{1}{(1-\lambda_k)\rho_k}(\hat{F} - E_k)P_X \tilde{x}_k +$$

$$\frac{1}{(1-\lambda_k)\rho_k}E_k P_X \tilde{x}_{k+1} + (1+\theta_k)\beta_2(x_{k+1}) \quad (18)$$

(1) 当 $e_{i_{k+1}} = \phi$ 时, 则 $e_{i_{k+1}}^T \phi = 1$, 所以

$$E_{k+1}\phi = \frac{\phi y_k^T + y_k \phi^T}{\|S_k\|}\phi - \frac{(y_k, \phi)}{\|S_k\|}\phi \phi^T \phi =$$

$$\frac{\phi y_k^T \phi + y_k}{\|S_k\|} - \frac{(y_k, \phi)\phi}{\|S_k\|} = \frac{y_k}{\|S_k\|} +$$

$$\frac{\phi y_k^T \phi - (y_k, \phi)\phi}{\|S_k\|} =$$

$$\frac{F(x_{k+1}) - F(x_{k+1} - \|S_k\|\phi)}{\|S_k\|} +$$

$$\frac{\phi y_k^T \phi - (y_k, \phi)\phi}{\|S_k\|} = \frac{\rho_k(1+\lambda_k)}{2}u +$$

$$\theta_k\beta_0(x_{k+1}) + (1+\tilde{\theta}_k)\beta_2(x_{k+1})$$

因此

$$D_{k+1}\phi = P_N E_{k+1}\phi = \frac{\rho_k(1+\lambda_k)}{2}\langle u, \phi \rangle \phi +$$

$$\theta_k\beta_0(x_{k+1}) + (1+\tilde{\theta}_k)\beta_2(x_{k+1})$$

$$B_{k+1}\phi = P_X E_{k+1}\phi = \frac{\rho_k(1+\lambda_k)}{2}P_X u +$$

$$\theta_k\beta_0(x_{k+1}) + (1+\tilde{\theta}_k)\beta_2(x_{k+1})$$

(2) 当 $e_{i_{k+1}} \neq \phi$ 时, $e_{i_{k+1}}^T \phi = 0$, 因此 $E_{k+1}\phi =$

$E_k \phi$, 则由式(18)得

$$D_{k+1}\phi = P_N E_{k+1}\phi = P_N E_k \phi = \frac{\rho_k}{2(1-\lambda_k)}\langle u, \phi \rangle \phi +$$

$$(1+\theta_k)\beta_2(x_{k+1}) + \frac{1}{(1-\lambda_k)\rho_k}P_N(E_k -$$

$$\hat{F})P_X S_k = \frac{\rho_k}{2(1-\lambda_k)}\langle u, \phi \rangle \phi +$$

$$\varepsilon_k\tilde{\theta}_k\beta_1(x_{k+1}) + (1+\theta_k)\beta_2(x_{k+1})$$

$$B_{k+1}\phi = P_X E_{k+1}\phi = P_X E_k \phi = \frac{\rho_k}{2(1-\lambda_k)}P_X u +$$

$$\frac{1}{(1-\lambda_k)}\hat{F}\tilde{x}_{k+1} + \frac{1}{(1-\lambda_k)\rho_k}P_X(E_k -$$

$$\hat{F})P_X S_k + (1+\theta_k)\beta_2(x_{k+1}) =$$

$$\frac{\rho_k}{2(1-\lambda_k)}P_X u + (1+\theta_k)\beta_2(x_{k+1}) +$$

$$(\varepsilon_k\tilde{\theta}_k + \theta_{k+1})\beta_1(x_{k+1})$$

上述两个引理的直接结果是

$$\tilde{D}_{k+1} = D_{k+1} - C_{k+1}A_{k+1}^{-1}B_{k+1} = D_{k+1} - \varepsilon_{k+1}\beta_1(x_{k+1})$$

可逆, 且 $\tilde{D}_{k+1}^{-1} = \beta_{-1}(x_{k+1})$

引理3 设引理1的结论成立, 则

$$P_N \tilde{x}_{k+2} = \frac{1}{1+\lambda_k}P_N \tilde{x}_{k+1} + (1+\tilde{\theta}_k)\beta_2(x_{k+1}) +$$

$$(\varepsilon_{k+1}\theta_{k+1} + \varepsilon_k\tilde{\theta}_k)\beta_1(x_{k+1}) \quad (19)$$

$$P_X \tilde{x}_{k+2} = (\varepsilon_{k+1}\theta_{k+1} + \varepsilon_k\tilde{\theta}_k)\beta_2(x_{k+1}) +$$

$$(1+\tilde{\theta}_k)\beta_3(x_{k+1}) \quad (20)$$

此引理的证明见文[1]中的引理2.30.

由式(18)得

$$\lambda_{k+1} = \frac{1}{1+\lambda_k} + \tau_k$$

其中 τ_k 是很小的正数, 所以当 $k = n+1$ 时(2.4)式成立.

由式(19)得:

$$x_{k+2} \in W(\rho, \theta_{k+2}, 2) \text{ 其中}$$

$$\begin{cases} \theta_{k+2} \leq c_2(\varepsilon_{k+1}\theta_{k+1} + \varepsilon_k\tilde{\theta}_k) \\ i_{k+1} = n \\ \theta_{k+2} \leq c_2(\varepsilon_k\tilde{\theta}_k + (1+\varepsilon_{k+1})\theta_{k+1} + \theta_k) \\ i_{k+1} \neq n \end{cases}$$

所以对 $k = n+1$ 时式(12)成立.

由引理1, 引理2, 引理3知, (12) ~ (15) 对一切正整数 $k \geq 0$ 均成立, 且(12) ~ (15) 暗含 $\lim_{k \rightarrow \infty} \|x_k - x^*\| = 0$ 和 $\theta_k \rightarrow 0$. 因此定理1的结论成立.

2 算 例

$$F(x, y) = \begin{pmatrix} x + y^2 \\ \frac{3}{2}xy + y^2 + y^3 \end{pmatrix}$$

其中 $n=2, x^*=(0,0)$

$$(x_0, y_0) = (0.005, 0.05)$$

$$E_0 = F(x_0, y_0), \lambda_k = \frac{\rho_{k+1}}{\rho_k}$$

计算结果如表 1 所示.

表 1 计算结果

k	x_k	y_k	λ_k
1	-0.232 6E-03	0.307 2E-01	0.614 312
2	-0.330 9E-05	0.204 5E-01	0.665 807
3	-0.825 7E-06	0.126 3E-01	0.617 432
4	-0.394 7E-07	0.781 4E-02	0.618 723
5	-0.998 7E-08	0.483 3E-03	0.618 476
6	-0.620 7E-09	0.299 7E-03	0.620 132
10	-0.553 1E-11	0.523 7E-06	0.616 478
20	-0.162 9E-16	0.723 9E-10	0.618 274

参 考 文 献:

- [1] DECKER W W, KELLEY C T. Broyden's Method For A Class of Problems Having Singular Jacobian At the Root [J]. SIMA. Numer Anal. 1985, 22: 555-574.
- [2] DECKER D W, KELLER H B and KELLEY C T. Convergence Rates for Newton's Method at Singular Points [J]. this Journal, 1983, 20: 296-314.
- [3] 金重方. 用列修正拟 Newton 法求解奇异问题 [J]. 黑龙江大学自然科学学报, 1990, 24-29.
- [4] 潘状元. 用修正的牛顿法求解奇异问题 [J]. 高等学校计算数学学报, 1988: 240-249.
- [5] 席少霖, 顾 明. 奇异点处的拟 Newton 方法 [J]. 计算数学, 1988, 291-298.

(编辑: 王 萍)

(上接第 63 页)

- [6] HU G D. Observers for One-sided Lipschitz Non-linear Systems [J]. IMA J. Math. Control Inf., 2006, 23: 395-401.
- [7] HU G D. A Note on Observer for One-sided Lipschitz Non-linear Systems [J]. IMA J. Math. Control inf., 2008, 25: 297-303.
- [8] IBRIR S. Circle-criterion Approach to Discrete-time Nonlinear Observer Design [J]. Automatica., 43: 1432-1441.
- [9] KAZANTZIS N, KRAVARIS C. Nonlinear Observers Design using Lyapunov's Auxiliary Theorem [J]. Systems and Control Letters., 1998, 34: 241-247.
- [10] KRENER A J, HEDRICK J K. Linearization by output Injection and Nonlinear Observers [J]. Syst. Control Lett., 1983(3): 47-52.
- [11] KRENER A J, RESPONDEK W. Nonlinear Observers Within-linearizable Error Dynamics [J]. SIAM J. Control Optim., 1985, 23: 197-216.
- [12] RAGHAVAN S, HEDRICK J K. Observer Design for a Class of Nonlinear Systems [J]. Int. J. Control., 1994, 59: 515-528.
- [13] RAJAMANI R. Observer for Lipschitz Nonlinear Systems [J]. IEEE Trans. Autom. Control, 1998, 43: 397-401.
- [14] RAJAMANI R, CHAO Y M. Existence and Design of Observers for Nonlinear Systems: Relation to Distance to Unobservability [J]. Int. J. Control, 1998, 69: 717-731.
- [15] THAU F E. Observing the State of Nonlinear Dynamic Systems [J]. Int. J. Control, 1973, 17: 471-480.
- [16] XU M Y, HU G D, ZHAO Y B. Reduced-order Observer Design for one-sided Lipschitz non-linear Systems [J]. IMA J. Math. Control Inf., 2009, 26: 299-317.
- [17] XIA X H, GAO W. Nonlinear Observer Design by Observer Error Linearization [J]. SIAM J. Control Optim., 2002, 27: 199-216.
- [18] ZHU F, HAN Z. A Note on Observers for Lipschitz Nonlinear Systems [J]. IEEE Trans. Autom. Control, 2002, 47: 1751-1754.
- [19] ZEMOUCHE A, BOUTAYEB M, BARA G I. Observers for a Class of Lipschitz Systems with Extension to H_∞ Performance Analysis [J]. System & Control Letters, 2008, 57: 18-27.

(编辑: 王 萍)